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THE PSEUDO-CIRCLE IS NOT HOMOGENEOUS

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In the first two volumes of *Fundamenta Mathematica*, Knaster and Kuratowski raised the following two questions [15], [16]: (1) If a nondegenerate, bounded plane continuum is homogeneous, is it necessarily a simple closed curve? (2) Does there exist a continuum each subcontinuum of which is indecomposable? Although Knaster settled the second question in 1922 [14], it was to remain until 1948 for the interrelationship of the two questions to become apparent. In that year Moise [18] described an indecomposable, chainable plane continuum which was homeomorphic to each of its proper subcontinua, and Bing [1] showed that this continuum was homogeneous. Bing [2] later showed that this continuum, which Moise called a pseudo-arc, was homeomorphic to Knaster's continuum.

In 1951 Bing [2] showed the existence of uncountably many topologically distinct, hereditarily indecomposable, plane continua. The continuum which he used as a tool in his construction was itself a promising candidate for inclusion in the class of homogeneous, hereditarily indecomposable, plane continua. This circularly chainable continuum had the property that, while it was topologically different from a pseudo-arc, every proper subcontinuum of it was a pseudo-arc; hence it seemed to combine some of the properties of the pseudo-arc and the simple closed curve. Such continua were later referred to as "pseudo-circles" and were discussed by F. B. Jones in [11].

In [20] the author discussed hereditarily indecomposable, circularly chainable continua in general and constructed uncountably many such continua, one of which had the additional property that it could be mapped continuously onto any circularly chainable continuum. The purpose of this paper is to prove that the pseudo-arc is the only homogeneous, hereditarily indecomposable, circularly chainable continuum (Theorem 2). It will first be shown that the pseudo-circle is not homogeneous (Theorem 1) and the methods will be adapted to prove the more general theorem.

C. E. Burgess [11] has asked whether each continuum, each proper subcontinuum of which is homogeneous, is itself homogeneous. Since each proper subcontinuum of a pseudo-circle is a pseudo-arc and hence is homogeneous, this paper answers Burgess' question in the negative.

In this paper a continuum is a nondegenerate, compact, connected subset of a metric space. A map or a mapping is a continuous, single-valued function. A

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continuum is indecomposable if it is not the union of two of its proper subcontinua. A continuum is hereditarily indecomposable if each of its subcontinua is indecomposable. A continuum is homogeneous if, for each pair of points in the continuum, there exists a homeomorphism of the continuum onto itself which takes one point into the other. A simple closed curve is the simplest example of a homogeneous plane continuum.

A chain [19] is a finite sequence $L_1, L_2, \ldots, L_n$ of open sets such that $L_i \cdot L_j \neq \varnothing$ if and only if $|i-j| \leqq 1$. If L_1 also intersects L_n, the sequence is called a circular chain. Each L_i is called a link. If ε is a positive number and if the diameter of each link is less than ε, then the sequence is called an ε-chain. A continuum is said to be chainable if, for each positive number ε, the continuum can be covered by an ε-chain. Circularly chainable continua are defined similarly. A pseudo-circle is defined to be a circularly chainable, hereditarily indecomposable, plane continuum which is not chainable. The pseudo-arc [1] is an example of a homogeneous, hereditarily indecomposable, plane continuum which is both circularly chainable and chainable. A solenoid is an example of a homogeneous, indecomposable continuum which is circularly chainable but not chainable.

Independent proofs that the pseudo-circle is not homogeneous were announced simultaneously by the author and L. Fearnley (see Notices Amer. Math. Soc. **15** (1968), p. 942 and p. 943). Fearnley's paper has appeared [7].

1. **Degrees of mappings.** Let C denote the circle in the plane with center at the origin and circumference two. Orient C so that a definite sense of positive rotation exists. Define the distance between two points of C to be the minimum of the lengths of the two arcs into which these points divide C. Let d denote this distance function. Note that $d(C)=1$, where $d(C)$ is defined to be

$$\sup \{d(x, y) \mid x \in C, y \in C\}.$$

Let A be an arc contained in C and let x be an endpoint of A. Call x the *initial point of A* if x is the least point of A with respect to the ordering of A induced by positive rotation. Call x the *terminal point of A* if x is the greatest point of A with respect to the ordering induced by positive rotation. If y is the initial point of A and z is the terminal point of A, then A may also be denoted by $[y, z]$, a symbol which has the advantage of indicating an ordering as well as a point set. Note that $[y, z]$ and $[z, y]$ are different arcs. The symbol $[y, y]$ is another name for C.

Let f be a map of C onto C. Associated with the map f is an integer, called the degree of f, which is intuitively the winding number of f or the number of times and the sense of rotation that $f(x)$ rotates around C as x makes one positive rotation. See [6, p. 335] for a rigorous definition of degree and for a proof of

LEMMA 1. *If f and g are two mappings of C onto C, then*

$$\text{degree}\,(f \circ g) = (\text{degree}\,f)\cdot(\text{degree}\,g).$$

Suppose that f and g are two mappings of C onto C and that ε is a positive number. Define the symbol $f =_\varepsilon g$ to mean that $d[f(x), g(x)] \leqq \varepsilon$, for each x in C.

LEMMA 2. *Let f and g be two mappings of C onto C, and let ε be a positive number which is less than one. Then $f =_\varepsilon g$ implies that the degree of f equals the degree of g.*

Proof. Since $d(C)=1$ and $\varepsilon<1$, $f(x)$ and $g(x)$ are not antipodal points, for any x in C. Hence [6, p. 316] f and g are homotopic. Then by the theorem of Hopf [6, p. 352], the degree of f equals the degree of g. $\square$

Let U be the class of maps of C onto C which have the property that if f is in U and x is a point of C, then $f^{-1}(x)$ has only a finite number of components. Let p be a point in C and let f be an element of U. Let $z_0, z_1, \dots, z_k$ be the boundary points of $f^{-1}(p)$, ordered by positive rotation. Let z_{k+1} be another name for z_0. Recall that for each i, $0 \leqq i < k$, $[z_i, z_{i+1}]$ denotes the arc of C such that z_i is the initial point when ordered by positive rotation. If $f|[z_i, z_{i+1}]$ is surjective, call $[z_i, z_{i+1}]$ an $A+$ or $A-$ according as the image of $[z_i, z_{i+1}]$ under f emanates from p in the positive or negative direction. If $f|[z_i, z_{i+1}]$ is not surjective, call $[z_i, z_{i+1}]$ a B. The following lemma, proved in [20], relates these notions to the concept of degree:

LEMMA 3. *Let f belong to U. Then the degree of f equals the number of $A+$'s of f diminished by the number of $A-$'s of f. Hence this number is independent of the point p.*

Lemma 3 motivates the following definition: If $[x_1, x_2]$ is an arc in C such that $f(x_1)=p=f(x_2)$, then the *degree of* $[x_1, x_2]$ *with respect to* f is the number of $A+$'s of $f|[x_1, x_2]$ diminished by the number of $A-$'s of $f|[x_1, x_2]$. Equivalently, by identifying x_1 with x_2, $f|[x_1, x_2]$ may be regarded as a map of a circle into a circle. The degree of $f|[x_1, x_2]$, regarded as a map of a circle into a circle, is the same as the degree of $[x_1, x_2]$ with respect to f. An obvious, yet important, fact is that Lemma 2 also holds for such restricted mappings with this generalization of degree.

LEMMA 4. *Let f and g belong to U, and let ε be a positive number which is less than one. Let $[x_1, x_2]$ be an arc in C such that $f(x_1)=f(x_2)=g(x_1)=g(x_2)$. Then $f|[x_1, x_2] =_\varepsilon g|[x_1, x_2]$ implies that the degree of $[x_1, x_2]$ with respect to f equals the degree of $[x_1, x_2]$ with respect to g.*

Also following immediately from the definition is the proof of

LEMMA 5. *Let f belong to U, and let a, b, and c be points of C such that $f(a) = f(b) = f(c)$. Then with respect to f*

$$\text{degree } [a, b] + \text{degree } [b, c] = \text{degree } [a, c].$$

Let C_1 and C_2 be two copies of C, and let $f\colon C_1 \to C_2$ be an element of U. The definition of $A+$, $A-$, and B partitions the domain of f into a finite collection $W(f, p)$ of A's and B's. Let $V=\{v_1, v_2, \dots, v_n\}$ be an enumeration of the elements of $W(f, p)$ such that the order of V agrees with positive rotation. For $1 \leqq i \leqq n$, $\{v_1, v_2, \dots, v_i\}$ is called an *initial segment of* V, and the arc in C_1 which is the union of $v_1, v_2, \ldots, v_i$ is called an *initial segment of* C_1. Similarly $\{v_i, v_{i+1}, \ldots, v_n\}$ is called a *terminal segment of* V and the corresponding arc in C_1 is called a *terminal segment of* C_1. The degree of an initial or terminal segment of V is defined to be the degree of the corresponding arc in C_1. V is said to be a *canonical ordering of* $W(f, p)$ if the degree of any initial segment of V is positive. Note that if V is a canonical ordering and if the degree of f is one, then the degree of any terminal segment is nonpositive. The initial point of v_1 is said to be an *initial point of* C_1 with respect to f and p.

LEMMA 6. *If f belongs to U and if degree $f=1$, then there exists an unique canonical ordering of $W(f, p)$.*

Proof. That there exists a canonical ordering of $W(f, p)$ was proven in [20, Lemma 8]. Suppose there exist two canonical orderings $V=\{v_1, v_2, \ldots, v_n\}$ and $V'=\{v_i, v_{i+1}, \ldots, v_n, v_1, \ldots, v_{i-1}\}$. Then $\{v_1, v_2, \ldots, v_{i-1}\}$, being an initial segment of V, has degree greater than or equal to one; in addition, $\{v_i v_{i+1}, \ldots, v_n\}$, being an initial segment of V', has degree greater than or equal to one. Since the degree of f equals the degree of $\{v_1, v_2, \ldots, v_{i-1}\}$ plus the degree of $\{v_i, v_{i+1}, \ldots, v_n\}$, the degree of $f \geqq 2$. This contradicts the hypotheses of the lemma. $\square$

LEMMA 7. *Let f and g be maps with positive degrees which belong to U, and let r belong to C. Let q be an initial point of a canonical ordering of $W(f, r)$, and let p be an initial point of a canonical ordering of $W(g, q)$. Then p is an initial point of a canonical ordering of $W(f \circ g, r)$.*

Proof. Let $z_0, z_1, \ldots, z_k$ be the boundary points of $[f \circ g]^{-1}(r)$, ordered by positive rotation. Let z_{k+1} be another name for z_0. Since p is one of the z_i's, suppose, without loss of generality, that p is z_0. The point p is an initial point of an ordering of $W(f \circ g, r)$. If this ordering is not a canonical ordering of $W(f \circ g, r)$, then there exists a smallest positive integer i such that $[z_0, z_i]$ has nonpositive degree with respect to $f \circ g$. Let j be the smallest integer greater than or equal to i such that z_j belongs to $g^{-1}(q)$. Such an integer exists and is not greater than $k+1$. By virtue of the fact that p is an initial point of a canonical ordering of $W(g, q)$, $[z_0, z_j]$ has positive degree with respect to g. Hence z_i and z_j are different points. By Lemma 1, the degree of $[z_0, z_j]$ with respect to $f \circ g$ is the product of the degree of f and the degree of $[z_0, z_j]$ with respect to g. Hence the degree of $[z_i, z_j]$ with respect to $f \circ g$ is greater than or equal to the product of the degree of f and the degree of $[z_0, z_j]$ with respect to g. However, in order for $[z_0, z_j]$ to have positive degree with respect to g, $[z_i, z_j]$ must be mapped by g onto a terminal segment of the canonical ordering of which q is an initial point. Since any terminal segment of a canonical ordering has degree less than the degree of the map, the degree of $[z_i, z_j]$ with respect to $f \circ g$, which is equal to the degree of $[g(z_i), g(z_j)]$ with respect of f, is less than the degree of f multiplied by the degree of $[z_0, z_j]$ with respect to g. This contradiction shows that the ordering determined by the sequence $z_0, z_1, \ldots,$ z_k is a canonical order, and the lemma is proved. $\square$

2. **The pseudo-circle as an inverse limit.** Let P be a pseudo-circle defined as the common part of a sequence $\{D_i\}$ of circular chains which satisfy the following conditions for each positive integer i:

(1) Each element of D_i is the interior of a circle of diameter less than $1/i$.

(2) The closure of each element of D_{i+1} lies in an element of D_i.

(3) Each D_i covers P.

(4) If E_i is a proper subchain of D_i and if E_{i+1} is a subchain of D_{i+1} which is contained in E_i, then E_{i+1} is crooked in E_i.

(5) D_1 has at least eight links.

(6) Every three consecutive links of D_{i+1} are contained in a link of D_i, and

(7) D_{i+1} is the union of two chains, K_1 and K_2, such that K_1 runs through D_i three times in the positive direction, K_2 runs through D_i two times in the negative direction, the first link of K_1 is also the last link of K_2, the first link of K_2 is also the last link of K_1, no other link of K_1 intersects a link of K_2, and each of the end links of K_1 is contained in only one (and the same) link of D_i.

Roughly speaking, (7) says that D_{i+1} runs through D_i three times in the positive direction and back two times in the negative direction. Hence D_{i+1} circles one time in D_i. According to Bing [5, Theorem 4], P is embeddable in the plane. Conditions (1)–(4) are usual conditions in defining a pseudo-circle; conditions (5)–(7) are imposed to simplify the constructions of this paper. Since the pseudo-circle is unique [8], we lose no generality in assuming these conditions.

Let k_i be the number of links of D_i. Denote the links of D_i by $d_1^i, d_2^i, \dots, d_{k_i}^i$. Let X_i denote C with k_i division points, $x_1^i, x_2^i, \dots, x_{k_i}^i$, dividing C into equal segments. Define a map π_i^{i+1} taking the division points of X_{i+1} into X_i by the following method: Let $\pi_i^{i+1}(x_j^{i+1})$ be x_t^i if d_t^i is the only link of D_i which contains d_j^{i+1}, and let $\pi_i^{i+1}(x_j^{i+1})$ be the midpoint of the arc the endpoints of which are x_t^i and x_{t+1}^i if d_j^{i+1} is contained in both d_t^i and d_{t+1}^i (there are, of course, two different arcs which have endpoints x_t^i and x_{t+1}^i; here we mean the arc which contains no x_s^i in its interior). Extend π_i^{i+1} linearly to map all of X_{i+1} onto X_i. Each π_i^{i+1} has degree one because of (7).

Define $\pi_i^n: X_n \to X_i$ by composition. Then $\{X_i, \pi_i^{i+1}\}$ forms an inverse sequence with limit X. Let $u_j^i = \pi_i^{-1}$ (open star of x_j^i), and let U_i denote the sequence $u_1^i, u_2^i, \dots, u_{k_i}^i$. Each U_i is a circular chain covering X.

The preceding construction and the following lemma are standard methods of passing from chain constructions to inverse limit representations (see, for example, [10, Lemma 1]).

LEMMA 8. *$X = \lim \{X_i, \pi_i^{i+1}\}$ is homeomorphic to P. Furthermore, the circular chains $\{U_i\}$ satisfy conditions* (1)–(7) *with respect to X.*

3. **Definition of two different points of X and proof that X is not homogeneous.** The next lemma, instrumental in defining different points of X, relies on (7) in the definition of the pseudo-circle.

LEMMA 9. *Let z be a division point of X_i. Then there exist three division points, x_1, x_2, x_3, of X_{i+1} such that $\pi_i^{i+1}(x_j)=z$, $j=1, 2, 3$, x_1 is the initial point of X_{i+1} with respect to $W(\pi_i^{i+1}, z)$*, degree $[x_1, x_2]=1$, degree $[x_2, x_3]=1$, *and* degree $[x_3, x_1]=-1$.

Proof. Consider the rectangle $Q=[0, 1]\times[0, 3]$ in the plane. Suppose that g is a map of $[0, 1]$ onto $[0, 3]$ which assumes integral values at zero and one. Define an equivalence relation on Q by $(x, y)\sim(x', y')$ if and only if $x=x'$ (mod 1) and $y=y'$ (mod 1). Then the equivalence relation associates the map g with a map of C onto C. Conversely, there exists a map h: $[0, 1]\to[0, 3]$ which is associated with π_i^{i+1} by this equivalence relation.

Let $z_1\in[0, 1)$, $z_2\in[1, 2)$, and $z_3\in[2, 3)$ be the three points of $[0, 3)$ which are associated with z under this equivalence relation. Let x_1' be the largest number in $[0, 1]$ such that $h(x_1')=z_1$. Let x_2' be the smallest number in $[0, 1]$ such that $x_2'>x_1'$ and $h(x_2')=z_2$. Let x_3' be the smallest number such that $h(x_3')=z_3$ and $x_3'>x_2'$. Numbers x_1', x_2', and x_3' exist because of condition (7) in the definition of the pseudo-circle.

Let x_1, x_2, and x_3 be the points of C which correspond to x_1', x_2', and x_3', respectively, under the equivalence relation. Since x_1' is the largest number in $[0, 1]$ such that $h(x_1')=z_1$ and since h is obtained from π_i^{i+1} which is obtained from a circular chain satisfying condition (7) in the definition of the pseudo-circle, x_1 is the initial point of X_{i+1} (a sketch of the graph of h helps one to see this point). It follows from the definition of x_1', x_2', and x_3' that degree $[x_1, x_2]=1$ and degree $[x_2, x_3]=1$. Since π_i^{i+1} has degree one, $[x_3, x_1]$ must have degree minus one. This proves the lemma. □

Two points of the pseudo-circle which are different will now be defined. Let y_1 be a division point of X_1. Having chosen $y_1, y_2, \ldots, y_k$, choose y_{k+1} to be the (unique) initial point of X_{k+1} with respect to π_k^{k+1} and y_k. The point $y=(y_1, y_2, \ldots)$ of X is called a *first point of X*. Let $z_1=y_1$. Having chosen $z_1, z_2, \ldots, z_k$, choose z_{k+1} as follows: There exist three points x_1, x_2, and x_3 in X_{k+1} which satisfy the conclusion of Lemma 9 (with $z_k=z$). Let $z_{k+1}=x_2$. The point $z=(z_1, z_2, \ldots)$ of X is called a *second point of X*.

THEOREM 1. *There does not exist a homeomorphism of X onto itself which carries z to y; therefore the pseudo-circle is not homogeneous.*

Proof. Suppose that there exists a homeomorphism f of X onto itself which carries z to y. If R is a circular chain covering X, let $f(R)$ denote the circular chain the links of which are images of the links of R under f. Choose positive integers n and m such that $f(U_n)$ refines U_1 and U_m refines $f(U_{n+1})$. That such integers m and n exist follows from a consideration of Lebesgue numbers and the fact that the homeomorphism f is uniformly continuous.

Define maps f_1' taking the division points of X_n into X_1 and f_2 taking the division points of X_m into X_{n+1} by a method similar to that by which the bonding maps π_n^{n+1} were defined: Let $f_1'(x_j^n)$ be x_t^1 if u_t^1 is the only link of U_1 which contains $f(u_j^n)$, and let $f_1'(x_j^n)$ be the midpoint of the arc of the endpoints of which are x_t^1 and x_{t+1}^1 if $f(u_j^n)$is contained in both u_t^1 and u_{t+1}^1 (again we mean the arc which contains no x_s^1 in its interior). Extend f_1' linearly to map all of X_n onto X_1. Define f_2 mapping X_m onto X_{n+1} similarly.

The maps f_1' and f_2 together with the bonding maps π_1^m and π_n^{n+1} form Diagram 1. If ε is a positive number, this diagram is said to be *ε-commutative* if $f_1'\pi_n^{n+1}f_2 =_\varepsilon \pi_1^m$. We now show that if ε is the length of a segment of X_1, then Diagram 1 is ε-commutative.

It is sufficient to prove ε-commutativity for the division points of X_m because all maps are extended linearly from the functional values at these points. Let x_i^m be a division point of X_m. The link u_i^m of U_m is contained in at most two links of U_1, say u_j^1 and u_{j+1}^1. This implies that $\pi_1^m(x_i^m)$ is contained in the arc of X_1 the endpoints of which are x_j^1 and x_{j+1}^1.

We must show that $f_1'\pi_n^{n+1}f_2(x_i^m)$ is contained in the same arc. U_m refines $f(U_{n+1})$ and hence refines $f(U_n)$. The map $\pi_n^{n+1}f_2$ carries x_i^m either into a division point p of x_n or into the interior of a segment $[p_1, p_2]$ of X_n, where p, p_1, and p_2 correspond to links of $f(U_n)$ containing u_i^m. Since $f(U_n)$ refines U_1, any link of $f(U_n)$ which contains u_i^m is contained in u_j^1 and/or u_{j+1}^1. Hence, in the former case, $f_1'(p)$ is contained in the arc of X_1 the endpoints of which are x_j^1 and x_{j+1}^1. In the latter case $f_1'(p_1)$ and $f_1'(p_2)$, and thus $f_1'([p_1, p_2])$, are contained in the arc of X_1 the endpoints of which are x_j^1 and x_{j+1}^1. This completes the proof that Diagram 1 is ε-commutative.

Now let u_j^n denote the link of U_n which contains z_n, and let u_t^1 denote the link of U_1 which contains y_1. The link $f(u_j^n)$ of $f(U_n)$ is contained in at least one link of U_1 because $f(U_n)$ refines U_1; since $f(u_j^n)$ contains $f(z)=y$ and since y is contained in only one link of U_1, the link $f(u_j^n)$ is contained only in u_t^1. Hence $f_1'(z_n)=y_1$. Similarly $f_2(y_m)=z_{n+1}$. This proves that $\pi_1^m(y_m)=f_1'\pi_n^{n+1}f_2(y_m)=y_1$.

Unfortunately, it is not necessarily the case that if p is a division point of X_m and if $\pi_1^m(p)=y_1$, then $f_1'\pi_n^{n+1}f_2(p)=y_1$. A slight modification of f_1', however, allows us to have this fact. Namely, suppose that p is a point of X_m such that $\pi_1^m(p)=y_1$ and $f_1'\pi_n^{n+1}f_2(p)\neq y_1$. We know that $f_1'\pi_n^{n+1}f_2(p)$ differs from y_1 by less then the length of a segment of X_1. Let $q=\pi_n^{n+1}f_2(p)$. We can modify the map f_1' slightly by "stretching" a small neighborhood Q of q so that the new map takes q to y_1. This neighborhood should be chosen so that it misses all division points of X_n except q and also misses the image under $\pi_n^{n+1}f_2$ of any other such point p in X_m. When this is done, note that we have not changed the image of any point in the complement of Q and that the image of any point in Q is moved by less than the length of a segment of X_1. We modify f_1' so that if $f_1'\pi_n^{n+1}f_2(p)=y_1$, then $\pi_1^m(p)=y_1$. If we perform this modification for each such p in X_m, the new map f_1 has the property that $\pi_1^m(p)=y_1$ if and only if $f_1\pi_n^{n+1}f_2(p)=y_1$. Furthermore

Lemma 2 implies that the degree of f_1 is the same as the degree of f_1'. Also Diagram 1, with f_1' replaced by f_1, is 2ε-commutative, where ε is the length of a segment of X_1. From now on, when we refer to Diagram 1, we shall always assume that f_1' has been replaced by f_1.

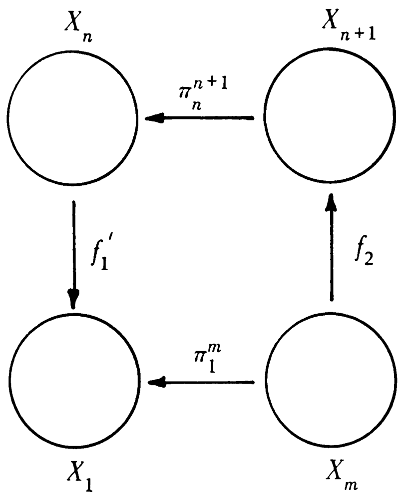

DIAGRAM 1

Since U_1 has at least eight links and since X_1 has circumference two, the length of a segment of $X_1 \leqq 1/4$. Since Diagram 1 is 2ε-commutative, where ε is the length of a segment of X_1, the maps π_1^m and $f_1\pi_n^{n+1}f_2$ satisfy the hypotheses of Lemma 2. Therefore the degree of π_1^m equals the degree of $f_1\pi_n^{n+1}f_2$. By construction, both π_1^m and π_n^{n+1} have degree one; accordingly, applying Lemma 1, either degree f_1 $=$ degree f_2, or degree $f_1 = -1 =$ degree f_2.

Let $D_1 = [d_1, z_{n+1}]$ be the minimal arc in X_{n+1} with z_{n+1} as its terminal point such that the degree of D_1 with respect to π_n^{n+1} and z_n is nonzero. Let $D_2 = [z_{n+1}, d_2]$ be the minimal arc in X_{n+1} with z_{n+1} as its initial point such that the degree of D_2 with respect to π_n^{n+1} and z_n is nonzero. By the manner in which z_{n+1} was chosen, z_{n+1} is the only point common to D_2 and D_1. Since both D_2 and D_1 are minimal arcs with nonzero degree, each has degree either plus one or minus one.

Let $E = [y_m, e]$ be the minimal arc of X_m with y_m as its initial point such that either $f_2(e) = d_1$ or $f_2(e) = d_2$. Then $f_1\pi_n^{n+1}f_2(e) \subset f_1\pi_n^{n+1}(\{d_1, d_2\}) = f_1(z_n) = y_1$. Hence E is an initial segment of X_m. Thus the degree of E with respect to π_1^m and y_1 is positive.

It is not possible to pick a point p in X_m such that p is not in E, $f_2(p) = z_{n+1}$, and degree $[y_m, p] = 0$ with respect to f_2; for this would mean that the degree of $[y_m, p]$ with respect to $f_1\pi_n^{n+1}f_2$ is zero. Since $f_2(p) = z_{n+1}$, $f_1\pi_n^{n+1}f_2(p) = y_1$. Hence $\pi_1^m(p) = y_1$ because Diagram 1 is commutative over $(\pi_1^m)^{-1}(y_1)$. Accordingly, $[y_m, p]$ is an initial segment of X_m with respect to π_1^m and y_1, and the degree of

$[y_m, p]$ with respect to π_1^m is positive. This contradicts Lemma 4 and proves the assertion in the first sentence of the paragraph. This assertion will be referred to as *Property* 1 in the sequel.

The remainder of the proof is divided into two cases. The first case is that degree $f_1=1=\text{degree } f_2$. In this case we show that $f_2(e)=d_2$.

Suppose $f_2(e)=d_1$. Let $q_1, q_2, q_3, \ldots, q_i, q_{i+1}=y_m$ be the division points of the arc $[e, y_m]$ such that $f_2(q_j)=z_{n+1}, j=1, 2, \ldots, i+1$. Assume the order of the sequence $\{q_i\}$ agrees with positive rotation. We now prove by induction that the degree of $[y_m, q_j]$ with respect to f_2 is negative $j=1, 2, \ldots, i+1$. We show the assertion is true for $j=1$. By Property 1, the arc $[y_m, q_1]$ does not have degree zero. In order that $[y_m, q_1]$ have positive degree, it is necessary that the arc $[e, q_1]$ be "wrapped" back around X_{n+1} in the direction of positive rotation. If this happened, some point in the interior of the arc $[e, q_1]$ must be mapped onto z_{n+1} by f_2. This contradicts the choice of q_1. Thus the degree of $[y_m, q_1]$ with respect to f_2 is negative.

Assume the assertion is true for $j=1, 2, \ldots, k$. Then the degree of $[y_m, q_k]$ with respect to f_2 is negative. The largest value which the degree of $[q_k, q_{k+1}]$ with respect to f_2 can take is plus one. Thus the degree of $[y_m, q_{k+1}]$ with respect to f_2 is either negative or zero. The degree of $[y_m, q_{k+1}]$ with respect to f_2 cannot be zero, by Property 1. Hence the degree of $[y_m, q_{k+1}]$ with respect to f_2 is negative. Hence the degree of f_2, which is equal to the degree of $[y_m, y_m]$ with respect to f_2, is negative. This contradicts the assumptions of Case 1 and completes the proof of the fact that $f_2(e)=d_2$.

Let x be the initial point of X_{n+1} with respect to π_n^{n+1} and z_n. Let $[y_m, b]$ be the minimal arc of X_m such that $f_2([y_m, b])\supset[z_{n+1}, x]$. By Property 1, x is the only common point of $f_2([y_m, b])$ and $[x, d_1]$. Being an initial segment of X_m with respect to π_1^m and y_1, $[y_m, b]$ has positive degree with respect to π_1^m. Hence, by Lemma 4, $[y_m, b]$ has positive degree with respect to $f_1\pi_n^{n+1}f_2$. The degree of $[y_m, b]$ with respect to $f_1\pi_n^{n+1}f_2$ equals the degree of $[z_{m+1}, x]$ with respect to $f_1\pi_n^{n+1}$. Being a terminal segment of X_{n+1} with respect to π_n^{n+1} and z_n, however, implies that $[z_{m+1}, x]$ has nonpositive degree with respect to π_n^{n+1} and hence with respect to $f_1\pi_n^{n+1}$. Therefore the degree of $[y_m, b]$ with respect to $f_1\pi_n^{n+1}f_2$ is nonpositive. This contradiction completes the proof that there do not exist mappings f_1 and f_2 with positive degree.

Assume now that the second case holds and that degree $f_1=-1=\text{degree } f_2$. By methods similar to those of the first case, we can show that if $f_2(e)=d_2$, then f_2 has nonnegative degree; hence $f_2(e)=d_1$.

We now show that the degree of D_1 with respect to π_n^{n+1} is plus one. Suppose that the degree of D_1 with respect to π_n^{n+1} is not plus one. Then, by the definition of D_1, the degree of D_1 with respect to π_n^{n+1} is minus one. Since f_1 has degree minus one, the degree of D_1 with respect to $f_1\pi_n^{n+1}$ is plus one. Since f_2 maps E onto D_1 in an order-reversing manner, the degree of E with respect to $f_1\pi_n^{n+1}f_2$ is minus one. E has positive degree with respect to π_1^m, however, because E is an initial segment of X_m with respect to π_1^m and y_1. This contradiction of Lemma 4 proves that the degree of D_1 with respect to π_n^{n+1} is plus one.

Let us review how we picked z_{n+1}. By Lemma 6, there exist three division points x_1, x_2, x_3, of X_{n+1} such that $\pi_n^{n+1}(x_j)=z_n, j=1, 2, 3$, x_1 is the initial point of X_{n+1} with respect to $W(\pi_n^{n+1}, z_n)$, degree $[x_1, x_2]=1$, degree $[x_2, x_3]=1$, and degree $[x_3, x_1]=-1$. We chose z_{n+1} to be x_2. Now let $v=x_3$. Then the degree of $[z_{n+1}, v]$ with respect to π_n^{n+1} is plus one.

Let $q_1=e, q_2, q_3, \ldots, q_{i+1}=y_m$ be the division points of the arc $[e, y_m]$ such that $\pi_n^{n+1}f_2(q_j)=z_n$, $j=1, 2, \ldots, i+1$. Assume the order of the sequence $\{q_i\}$ agrees with positive rotation. We can prove by an argument similar to the one used in Case 1 that the degree of $[y_m, q_j]$ with respect to $\pi_n^{n+1}f_2$ is negative for $j=1, 2, \ldots, i+1$.

Property 1 guarantees us that for no q_j does the arc $[y_m, q_j]$ have degree zero with respect to f_2. Hence let r be the smallest integer such that $f_2(q_r)=z_{m+1}$. Let s be the largest integer less than r such that $f_2(q_s)=v$. The degree of $[y_m, q_r]$ with respect to $\pi_n^{n+1}f_2$ is minus one. Hence the degree of $[y_m, q_r]$ with respect to $f_1\pi_n^{n+1}f_2$ is plus one. Accordingly, the degree of $[y_m, q_r]$ with respect to π_1^m is plus one. Therefore the degree of the terminal segment $[q_r, y_m]$ with respect to π_1^m is zero. Since f_2 maps $[q_s, q_r]$ onto $[v, z_{m+1}]$ in an ordering-reversing manner, the degree of $[q_s, q_r]$ with respect to $\pi_n^{n+1}f_2$ equals minus the degree of $[z_{m+1}, v]$ with respect to π_n^{n+1} equals minus one. Since the degree of f_1 is minus one, the degree of $[q_s, q_r]$ with respect to $f_2\pi_n^{n+1}f_1$ is plus one. Accordingly, by Lemma 4, the degree of $[q_s, q_r]$ with respect to π_1^m is plus one. Hence with respect to π_1^m,

$$\begin{aligned}\text{degree } [q_s, y_m] &= \text{degree } [q_s, q_r]+\text{degree } [q_r, y_m]\\ &= 1+0 = 1.\end{aligned}$$

Therefore the terminal segment $[q_s, y_m]$ has positive degree with respect to π_1^m. This contradiction completes the proof of the fact that there do not exist mappings f_1 and f_2 which have negative degrees and which complete Diagram 1.

Hence there do not exist mappings f_1 and f_2 with either positive or negative degrees. Therefore there is no homeomorphism of X onto itself taking z into y; the pseudo-circle, hence, is not homogeneous. □

We proceed now to a generalization of Theorem 1. The pseudo-arc is known to be a homogeneous and hereditarily indecomposable plane continuum. That it is circularly chainable can be easily deduced from the fact that it is chainable between each pair of its opposite end points [3]. The substance of the next theorem is that these properties are a characterizaton of the pseudo-arc. Before we prove the theorem, however, a trivial lemma is in order.

LEMMA 10. *If X is a circularly chainable continuum, then X is homeomorphic with the inverse limit of an inverse sequence $\{X_n, \sigma_n^{n+1}\}$, where each factor space X_n is a circumference and one of the following conditions holds for the bonding maps σ_n^{n+1}:*

(A) *The degree of each* σ_n^{n+1} *is zero.*
(B) *The degree of each* σ_n^{n+1} *is one.*
(C) *The degree of each* σ_n^{n+1} *is greater than one.*
(D) *The degree of each* σ_n^{n+1} *is minus one.*
(E) *The degree of each* σ_n^{n+1} *is less than minus one.*

Proof. One may easily see as a special case of results of J. R. Isbell [10] that a necessary and sufficient condition that X be circularly chainable is that X be homeomorphic to the inverse limit of an inverse sequence $\{X_n, \sigma_n^{n+1}\}$, where each factor space X_n is a circle and each σ_n^{n+1} is surjective. The lemma now follows from repeated applications of Lemma 1 together with the following fact about inverse limit sequences: If $\{Y_n, f_n^{n+1}\}$ is an inverse sequence with inverse limit Y, if $\{X_{m(n)}\}$ is a subsequence of $\{Y_n\}$, and if X is the inverse limit of the inverse sequence $\{X_n, f_{m(n)}^{m+1(n)}\}$, then X is homeomorphic to Y. □

THEOREM 2. *The pseudo-arc is the only homogeneous, hereditarily indecomposable, circularly chainable continuum.*

Proof. Let X be a homogeneous, hereditarily indecomposable, circularly chainable continuum. Then, by Lemma 10, X may be represented as the inverse limit of an inverse sequence $\{X_n, \sigma_n^{n+1}\}$, where each factor space X_n is a circle and the bonding maps σ_n^{n+1} satisfy one of conditions (A)–(E). Without loss of generality, we shall assume that each bonding map σ_n^{n+1} has nonnegative degree.

Suppose that each bonding map σ_n^{n+1} has degree zero. Ingram [9] has shown that if K is a circularly chainable continuum which is defined as the intersection of the circular chains $K_1, K_2, K_3, \ldots$, and if, for each natural number n, the circular chain K_{n+1} circles zero times in K_n (in the sense of [5]), then K is (linearly) chainable. This condition is equivalent to being the inverse limit of an inverse sequence where the factor spaces are circles and the bonding maps have degree zero. In this case, therefore, X is a homogeneous, chainable continuum. Bing [4] has shown that the only homogeneous, chainable continuum is the pseudo-arc.

Suppose that each bonding map σ_n^{n+1} has degree one. Bing [5] has shown that if K is a circularly chainable continuum which is defined as the intersection of the circular chains $K_1, K_2, K_3, \ldots$, and if, for each natural number n, the circular chain K_{n+1} circles one time in K_n, then K is embeddable in the plane. Once again this condition is equivalent to being the inverse limit of an inverse sequence where the factor spaces are circles and the bonding maps have degree one. In this case, therefore, X is planar and hence the pseudo-circle; thus, by Theorem 1, X is not homogeneous.

Finally, suppose that each bonding map σ_n^{n+1} has degree greater than one. Bing [5] has shown that such continua are not planar. Hence, in this case, X is one of the continua studied in [20]. Essentially the same argument as that of Theorem 1 may be employed to show that X is not homogeneous. The only difference in the proof is in the fact that an initial point with respect to the bonding map σ_n^{n+1} is not unique; for instance, if the degree of σ_n^{n+1} were two, then there would exist two points of X_{n+1} which would be initial points of σ_n^{n+1} with respect to some fixed point in X_n. The essential fact, however, is that there still exist first points and second points, and it is only necessary to pick a first point and a second point and to apply the proof of Theorem 1 (with appropriate modifications) to show that X is not homogeneous.

Therefore a continuum is a pseudo-arc if and only if it is homogeneous, hereditarily indeomposable, and circularly chainable. $\square$

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